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A Space-filling Three-dimensional Serial Polyaxis

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Abstract. A space-filling three-dimensional serial polyaxis analogous to the Peano-Hilbert curve, fractal recursion, the Euler path, and the Hamilton path is presented. A polyaxis is an object constructed by linear axes representing the object form and structural relationships, and is applied here to the construction of serial space-filling curves. A two-dimensional representation of a three-dimensional serial polyaxis is also devised. The construction is consistent with the graph theory and object-oriented representations of objects, and involves seriality and recursion. The representation can also be readily extended to computational geometry applications and information sciences. It is shown that a closed circuit is not possible in spaces constructed by an odd number of units, where start and end points of the space-filling serial polyaxis must both be located in odd-numbered units.

1. Introduction

One of the present authors has made the assertion that the order of life can be represented by a complex logarithmic spiral function based on the golden section, named the "golden spiral" (OKA, 1993). Geometric forms having 5-times rotational symmetry are common in life, such as the rhombic triacontahedral form of the capsid of virii. The capsid is constructed of chains of helical polypeptides, with each chain having four modules where the arm module forms a wireframe-like polyhedron. Proteins (i.e., polypeptides) the main building material of life, may take a spherical form in which part of the polypeptide forms an α helix. The actin filament has a double-helix form, while collagen assumes a triple-helix form. DNA, which carries the genetic code of all life, forms a double helix and the golden spiral. As a new representation for the construction of the golden form, inspired by the linear-segment construction of baskets, netting, embroidery, cloth, and even characters and line drawings, the concept of a "polyaxis" has been introduced. A polyaxis consists of linear axes with length, direction, and spin. The concept of a polyaxis and polyhedral forms

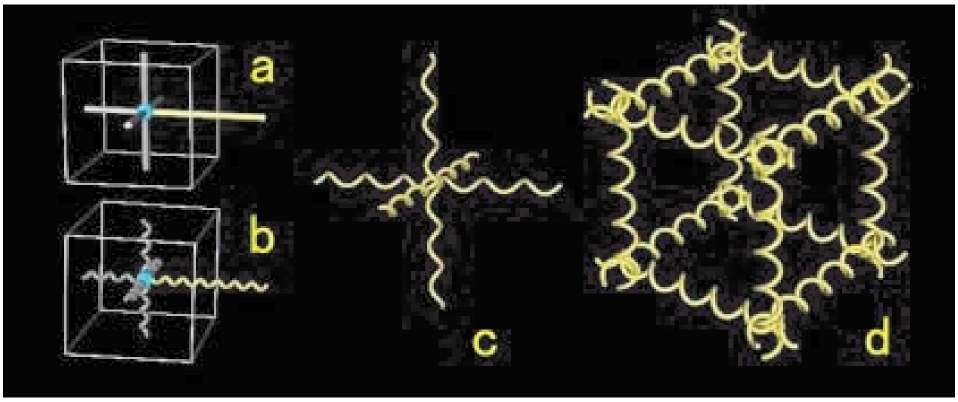

Fig. 1. (a) Unit axis. (b) Unit helical axis. (c) Orthogonal helical polyaxis. (d) Helical cubic polyaxis <4,4,4>. (c), (d) Separated crossing, where the blue point at the center of the unit cubic space represents the unit space and an object, equivalent to the mother point of a Volonoi cell or a node in graph theory. (a), (b) Yellow axis has unit length, direction, and spin. The axis represents a relationship, function, or connection, corresponding to an edge in graph theory. The axis also represents the point and the axis, i.e., an attribute, function, and relation, equivalent to an object-oriented approach.

such as linear polyaxes, combo polyaxes, orthogonal polyaxes, and twisted polyaxes have been reported (OKA, 2003), and the extension to pivot polyaxes, helical polyaxes, stellate polyaxes, rhombic polyaxes, and space-filling polyaxes has been discussed (OKA and KAWAMOTO, 2004, 2005a, 2005b, 2006a, 2006b).

In computational science, objects are managed by digits (zero-dimensional) and serial flow (one-dimensional). The Peano curve is a serial (one-dimensional) curve that fills two, three and more dimensional space. In a three-dimensional world where most objects have three-dimensional form, most life involves chirality. A spiral has intrinsic direction and chirality, suggesting that a space-filling three-dimensional serial polyaxis may provide a new representation of the golden spiral in life. A two-dimensional expression of a three-dimensional polyaxis is also devised and presented.

2. Definitions

Objects and relationships can be expressed in a number of ways via language, function, and form. Graph theory and object-oriented design are representative forms of expression. In the former, objects are expressed by a node (point) and a leaf (edge, segment of a line, arrow line), while in the latter, objects are expressed in terms of attributes and functions. The Peano-Hilbert curve, Euler path, and Hamilton path are examples of serial curves that can be drawn with one stroke. These curves and paths are applicable to a space-filling cuboidal space. The concepts involved in this representation of space-filling three-dimensional serial polyaxes are graph theory, object-oriented representation, space-filling curves, and the space-filling polyhedral unit (Figs. 1a and b). Here, an axis is defined as a linear feature representing an object and/or relationship, with intrinsic length, direction, and spin (see Figs. 1a and b), a polyaxis is the object or form composed by such axes, and a serial polyaxis is an object consisting of serially connected polyaxes in either an open (one way) or closed (circuit) configuration. The unit consists of a space (object), a representative point, and an axis (Figs. 1a and b), which has intrinsic direction and spin, and thus chirality with dextral or sinistral sense following the right-screw rule (right-hand theory) (Figs. 1b–d). An orthogonal axis has chirality, and may be closed or open (separated crossing) (Figs. 1c and d).

3. Construction and Expression

A space-filling three-dimensional serial polyaxis is constructed by serial connection of units (cube and axis) (Fig. 2; 2 × 2 × 2, 3 × 3 × 3, 3 × 5 × 3, 2 × 3 × 3). A two-dimensional polyaxis representation of a three-dimensional polyaxis can be constructed by slicing the three-dimensional polyaxis and turning the even-numbered of the serially numbered (1, 2, 3, ...) layers upside down for connection with the bottom of the immediate anterior layer (Fig. 2; 2 × 2 × 2b–e → 4 × 2B–E, 3 × 3 × 3b–c → 9 × 3b–c, 3 × 5 × 3b → 9 × 5b, 2 × 3 × 3b → 6 × 3b). Any space-filling rectangular parallelepiped serial polyaxis can be constructed and expressed by this method (Fig. 2; 3 × 5 × 3b → 9 × 5b, 2 × 3 × 3b → 6 × 3b).

Space-filling serial polyaxes are divided into even and odd types according to the numbers of units involved. Examples of even types are 2 × 2 × 2, 4 × 2, 4 × 4 × 4, 2 × 3 × 3, and 6 × 3, while odd types are 3 × 3 × 3, 9 × 3, 3 × 5 × 3 (Fig. 2). In even types, the start and end points lie in opposite odd/even or even/odd units, while in odd types, the start and end points are both located in odd-numbered units. A closed (circuit) type is composable in even types (Fig. 2; 4 × 2B, 2 × 2 × 2b, 6 × 3b, 2 × 3 × 3b), whereas all odd types of space-filling serial polyaxis are open (one way) (Fig. 2; 3 × 3 × 3b–c, 9 × 3b–c, 3 × 5 × 3b, 9 × 5b).

4. Serial and Recursive Serial Polyaxes

4.1. Serial polyaxis <3*3*3(l, m, n)–(p, q, r)>

A space [3 × 3 × 3] and the corresponding two-dimensional representation [9 × 3] consist of 27 units (i.e., odd type), and the start and end points of the space-filling serial polyaxis are located in odd-numbered units (Fig. 3; 3 × 3 × 3, 9 × 3). All of such space-filling polyaxes <3*3*3> are open (Fig. 3; <A>–<F>). Examples of space-filling serial polyaxes of the form <3*3*3(*l*, *m*, *n*)–(*p*, *q*, *r*)> are presented in Fig. 3 (<A> (–1, 1, 1)–(0, 1, 0), <B> (0, 0, 1)–(0, 0, –1), <C> (–1, 1, 1)–(0, 0, –1), <D> (–1, 1, 1)–(–1, 1, –1), <E> (–1, 1, 1)–(1, 1, –1), <F> (–1, 1, 1)–(1, –1, –1)).

4.2. Recursive Serial Polyaxis <<3*3*3>*2*2*2>

A recursive serial polyaxis <<Module>*2*2*2> is constructed by dividing a unit into 8 spaces and obtaining a reduced module <3*3*3> in each space (Fig. 3; <<M>*2*2*2> for M = <A>, <C>–<F>). The modules are then connected serially according to the pattern of the serial polyaxis of <2 × 2 × 2> showed in Fig. 2. From other point of view, recursive module is constructed with substituting the module "M" for the eight vertices of the mother connecting pattern of <2 × 2 × 2>. In the case of closed serial module <<M>*2*2*2>, it

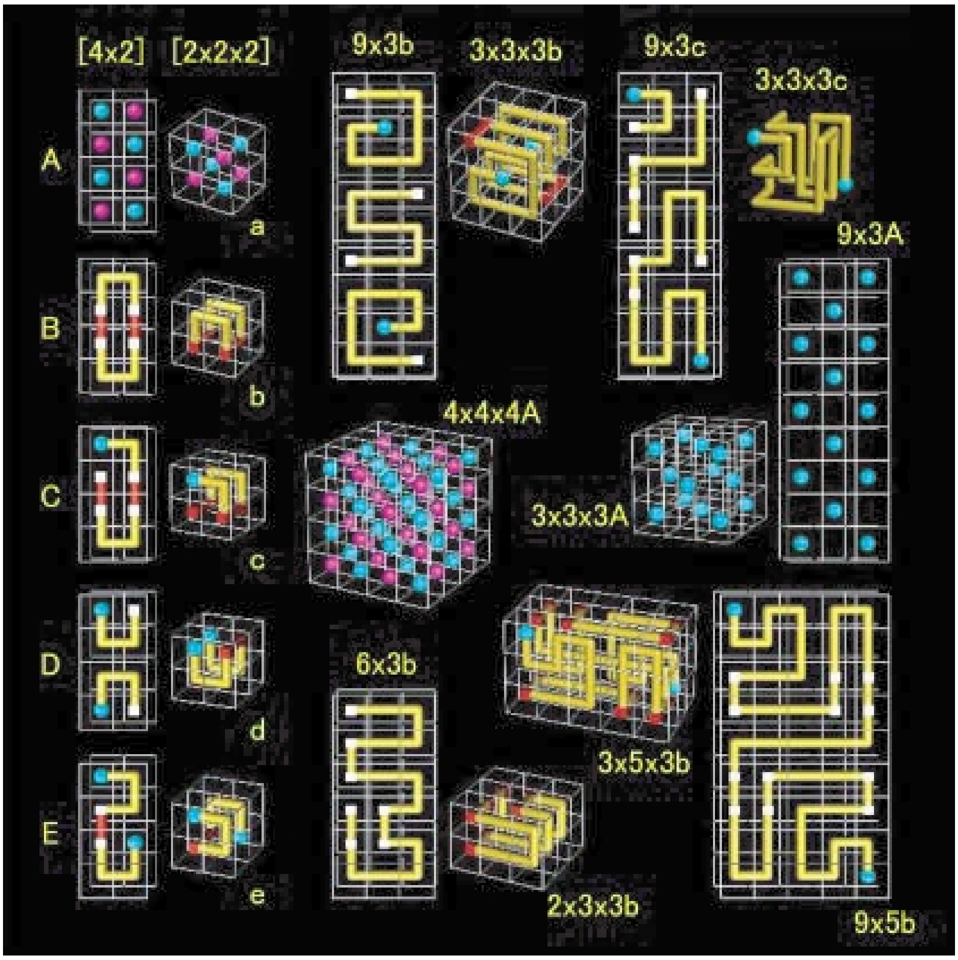

Fig. 2. Construction and expression. Blue and pink spheres denote start and end points, respectively. White and red cubes denote start or end points of antero-posterior connected axes.

becomes the next opened serial module by opening at the vertex. And the next serial module <<<M>2∗2∗2>2∗2∗2> is made, substituting the opened serial module <<M>∗2∗2∗2> for the eight vertices of the mother connecting pattern of <2 × 2 × 2> showed in Fig. 2 (Fig. 8; e). This is a recursive method, analogous to construction of a Peano-Hilbert curve or a fractal.

In the module polyaxis <3∗3∗3(0, 0, 1)–(0, 0, –1)> (Fig. 3B), the start and end points are located oppositely with respect to the face-centered cubic structure. Serial connection of this module therefore constructs straight serial chains of the modular polyaxis (Fig. 3; <<M>∗1∗1∗2>). Curves are possible in these straight chain cases by connecting other modules, such as the polyaxis <3∗3∗3(0, 0, 1)–(1, 0, 0)>.

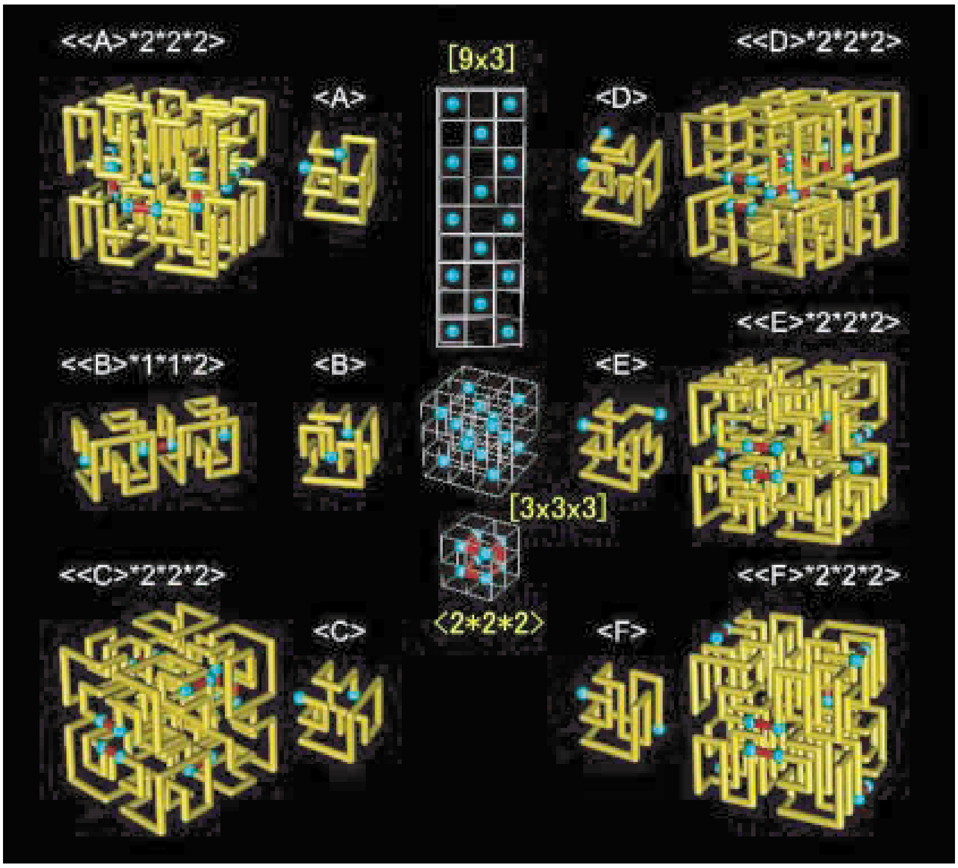

Fig. 3. Serial polyaxis <3*3*3> and recursive serial polyaxis <<Module>*2*2*2> for modules <A> to <F>. Spheres in odd-numbered units in [9 × 3] and [3 × 3 × 3] are start/end points of the space-filling serial polyaxis. The space-filling serial polyaxis <2*2*2> is a mother pattern for a recursive serial polyaxis.

5. Serial Polyaxis by Nested Combination

A nested combination of <<2*2*2>/<3*3*3>> serial polyaxis can be constructed such that a <2*2*2> serial polyaxis is obtained in a <3*3*3> serial polyaxis, where the start and end points are connected (Fig. 4; IIIa + IIIb → IIIc). A nested combination of <<2*2*2>/<3*3*3>> serial polyaxes can also be constructed from a <<4*4*4>2*2*2> serial polyaxis as a <<3*3*3>3*3*3> serial polyaxis with start and end points connected (Fig. 4; IIe → IIIe ← Ie). This approach is applicable for any nested combination of serial polyaxes <<p*q*r>/<l*m*n>>. Any combination of modules is possible, affording a nested combination serial polyaxes. For example, there are 6 patterns for the <3*3> module (Fig. 4; Ia).

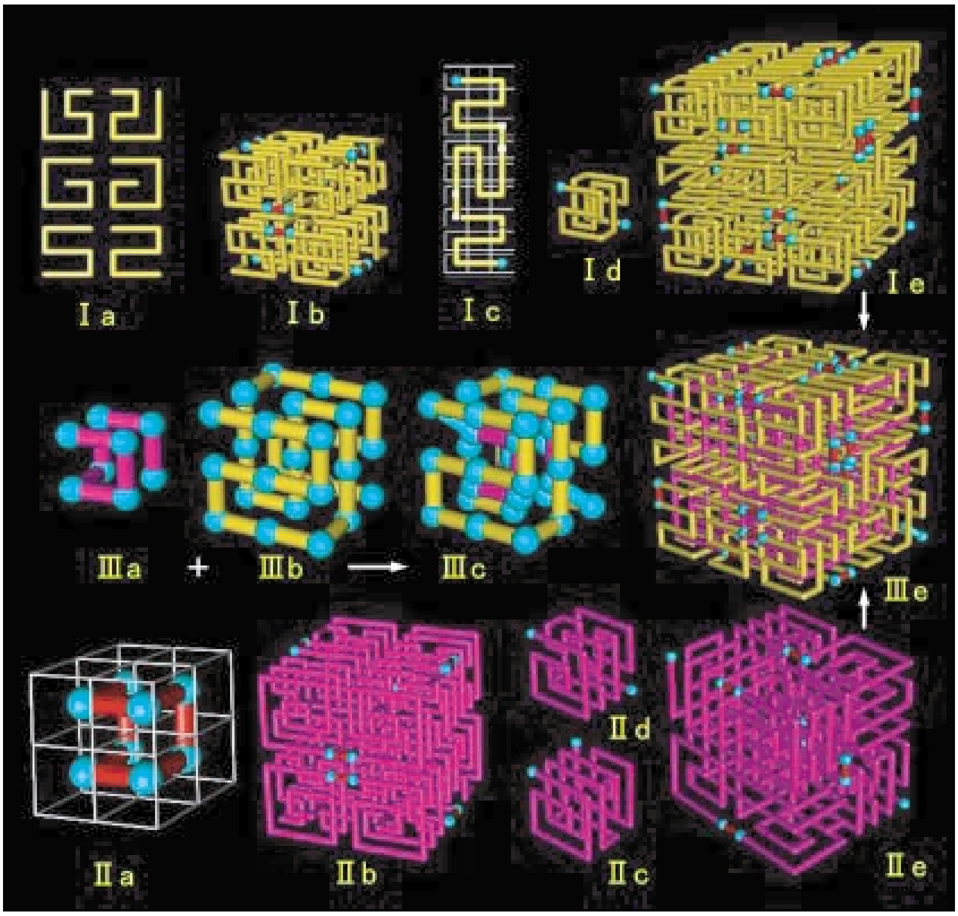

Fig. 4. Serial polyaxis by nested combination. Ia, <3*3>; Ib, <<3*3*3>*2*2*2>; Ic → Id, <3*3*3>; Ie, <<3*3*3>*3*3*3>; IIa, <2*2*2close>; IIb, <<4*4*4>*2*2*2>; IIc, <4*4*4(1, 1, 1)–(1, 1, 4)>; IId, <4*4*4(1, 1, 1)–(4, 4, 4)>; IIe, <<Iic + IId>*2*2*2>; IIIa, <2*2*2(1, 1, 1)–(2, 2, 2)>; IIIb, <3*3*3(1, 1, 1)–(3, 3, 3)>; IIIc, Combo<IIIa/IIb>; IIIe, Combo<IIe/Ie>.

6. Serial Polyaxis <<5*5*5>*p*q*r>

A <<5*5*5>**p*q*r*> serial polyaxis is constructed by a recursive method. Examples of the symmetrical patterns of a serial polyaxis are presented in Fig. 5 (<5*5(1, 1)–(5, 5)>). The modular serial polyaxis <5*5*5(1, 1, 1)–(5, 5, 5)> is constructed using patterns of a <5*5(1, 1)–(5, 5)> serial polyaxis. Similarly, a <<5*5*5(1, 1, 1)–(5, 5, 5)>*2*2*2> serial polyaxis is constructed using the <5*5*5(1, 1, 1)–(5, 5, 5)> module polyaxis and a serial pattern of <2*2*2closed> serial polyaxes. A <<5*5*5(1, 1, 1)–(5, 5, 5)>*3*3*3> serial polyaxis can be constructed using <5*5*5(1, 1, 1)–(5, 5, 5)> module polyaxes and a serial pattern of <3*3*3(1, 1, 1)–(3, 3, 3)> serial polyaxes (Fig. 5).

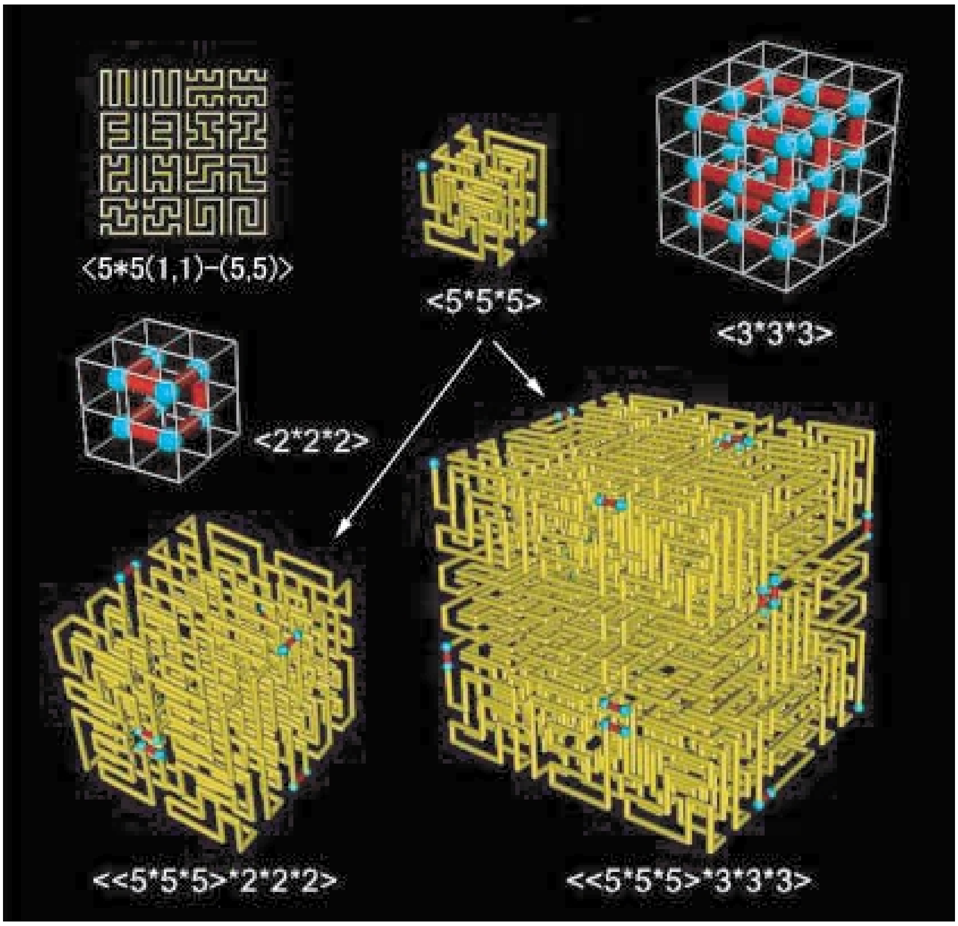

Fig. 5. Symmetrical patterns of serial polyaxis <5∗5∗5> as a module for recursive construction of <<module>∗2∗2∗2> or <<module>∗3∗3∗3>, where <2∗2∗2> and <3∗3∗3> are connecting patterns for recursion. The serial polyaxis <<module>∗2∗2∗2> or <<module>∗3∗3∗3> becomes the next module of a higher-level serial polyaxis.

7. Knots

A single knot, square knot, and vertical knot can be expressed by many forms (Fig. 6), but all forms are a type of serial polyaxis. The square knot and vertical knot are constructed using the cubic method (Fig. 6; Sc, Vc). A single knot is formed as a space-filling serial polyaxis of type <5∗5∗1 (–1, 2, 0)–(–1, 1, 0)> (Fig. 6; left half of the Sc, Vc), which is an odd type ($5 \times 5 \times 1 = 25$). The single knot has three crossings (three more spaces), causing the odd-numbered spaces to become even-numbered, starting from an even/odd numbered space (–1, 1, 0)/(–1, 2, 0) and ending at an odd/even numbered space (–1, 2, 0)/(–1, 1, 0)

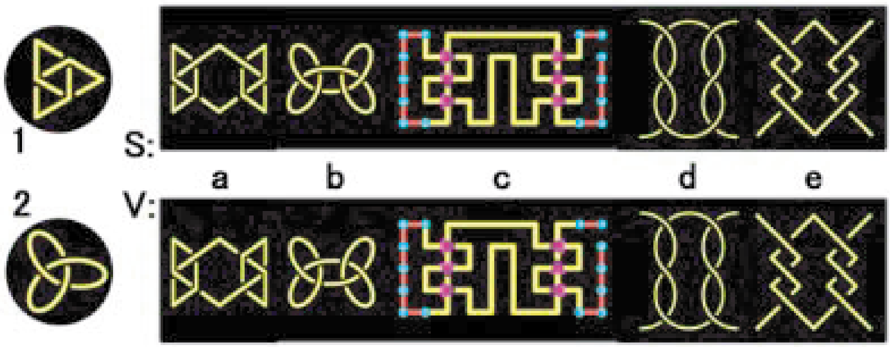

Fig. 6. Form of Single knot, Square knot, and Vertical knot: (1, 2) Closed single knot. (S) Square knot. (V) Vertical knot. (a), (b) Closed serial knot. (c) An expression followed cubic method. A serial knot expressed in <5∗10∗1>. Arrow head denotes separate crossing and direction. Red edge denote connecting axis. Blue point denotes start and/or end point of connecting axis. (d), (e) Core part of knot.

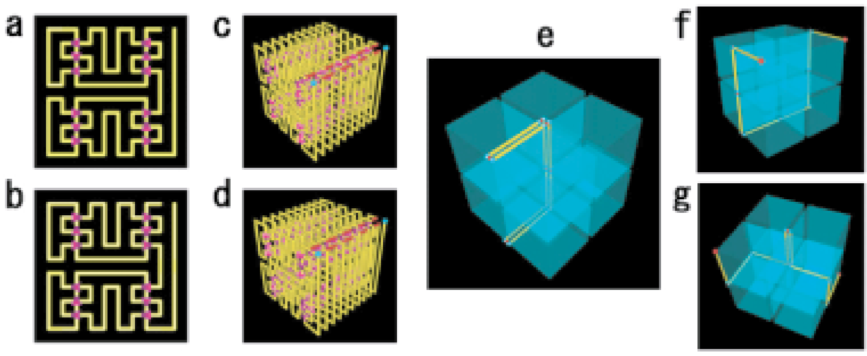

Fig. 7. Square knot and Serial recursive module: (a) Forward module<10∗10∗1>, starting point locates at vertex. (b) Backward module, ending point locate at vertex. (c) Forward module, constructed with connecting a-module and b-module alternately. (d) Backward module, connected b-module and a-module alternately. (e)–(g) Recursive module, c or d-module substituted for eight vertices of mother pattern of <2 × 2 × 2> showed in Fig. 2. Blue box denotes a module, and yellow edge denotes seriality of module. Blue point denotes a start or an end point.

(Fig. 6; left half of the Sc, Vc).

A square knot has orthogonal (reflectional) symmetry and chirality (Fig. 6; Sa–e), whereas a vertical knot has translational (gliding) symmetry or rotational symmetry (Fig. 6; Va–e).

A recursive serial polyaxis is made with knots, according to the method discussed at the fourth section. A $10 \times 10 \times 1$ module of square knot is made with connecting two square

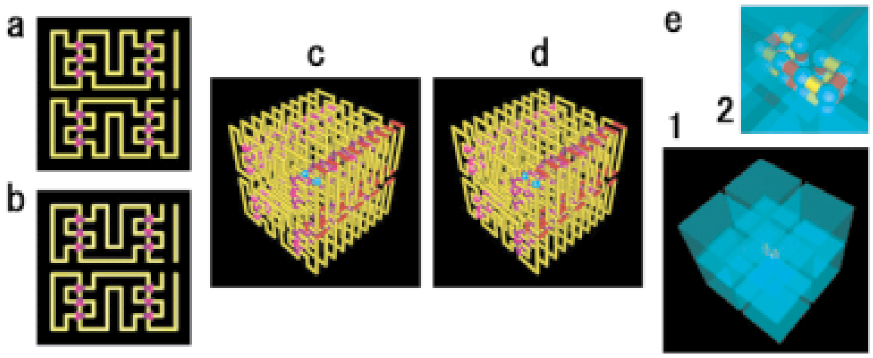

Fig. 8. Vertical knot and Serial recursive module: (a) Forward module <10*10*1>. (b) Backward module. (c) Forward module <10*10*10>, constructed with arranging a-module and b-module alternately, and firstly upper half of module connected antero-posteriorly, secondly lower half of module connected postero-anteriorly. (d) Backward module, b-module and a-module arranged alternately. (e1, 2) A recursive module <<c or d>2*2*2>, c or d-module substituted for vertices of closed pattern of $<2 \times 2 \times 2>$ showed in Fig. 2. Blue box denotes a module, and yellow edge denotes seriality of the module. Blue point denotes a start or an end point. Recursive-ness is kept by opening a vertex of the closed module.

knots (Fig. 7; a and b). In this paper, it is called a "forward" module that starting point located in the vertex (Figs. 7 and 8; a and c), and "backward" module that ending point located in the vertex (Figs. 7 and 8; b and d). A $10 \times 10 \times 10$ module of knot is made with connecting the forward and backward modules alternately (Figs. 7 and 8; c and d). A next module $<<10 \times 10 \times 10>2*2*2>$ is made with substituting the $10 \times 10 \times 10$ module for the eight vertices of the mother connecting pattern of $<2 \times 2 \times 2>$ showed in Fig. 2 (Fig. 7; e–g, Fig. 8; e).

8. Discussion

The existence of a closed Hamilton path in the general graph is an NP-complete problem. From the results above, the condition for a closed Hamilton path within the proposed framework is that the space must consist of an even number of units, that is, a closed path is not possible in odd-type spaces.

This method for forming a serial polyaxis is recursive and nested, and the result has self-similar form. Therefore, the same pattern appears repeatedly in the serial polyaxis, and module objects or connection patterns in the serial polyaxis are exchangeable. According to this construction, a variety of serial polyaxes can be composed at will, affording three-dimensional Fisher-like pictures or arabesque patterns. The proposed construction has seriality, recursion, and a correspondence between 3D/2D objects and 1D/0D objects. The object can therefore be readily extended to computer-based geometric applications (e.g., computer graphics, design, integrated circuits) and information sciences (e.g., algorithms, cryptography).

9. Conclusion

A space-filling three-dimensional serial polyaxis analogous to the Peano-Hilbert curve, fractal recursion, the Euler path, and the Hamilton path was presented. The representation of objects in the scheme is consistent with graph theory and object-oriented approaches, and allows a three-dimensional space-filling serial polyaxis to be represented by a two-dimensional expression. The proposed approach involves seriality and recursion, and provides correspondence between 3D/2D objects and 1D/0D objects. It was shown that in odd-numbered spaces, the start and end points of a space-filling serial polyaxis are located in odd-numbered units, and that the serial polyaxis cannot form a circuit. For a closed Hamilton path to exist, it was determined that the space must be constructed by an even number of units. A single knot, square knot, and vertical knot can be expressed by this method, and examples of three-dimensional combinations of knots were composed. This construction can thus be used to produce three-dimensional fractal patterns, arabesque patterns, or Fisher-like pictures.

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