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Knots, Links, and Self-Avoiding Curves

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Abstract. This paper considers different mirror curves with the maximal number of mirrors, resulting in self-avoiding curves. They are enumerated and represented by corresponding chord diagrams. Their connection with knot theory is also considered.

This work is inspired by a series of sculptures titled *Viae Globi*, created by Carlo Sequin (Fig. 1) (SEQUIN, 2001), and by a conversation with Haresh Lalvani, who proposed to collapse vertices of a polygon, in particular two vertices of a triangle in order to obtain "a triangle with two vertices" (Fig. 2). This simple idea is a part of his more extended unpublished work. Generalizing this idea, we can conclude that every knot or link shadow with n crossings is an $2n$-gon with n pairs of collapsed points.

The paper deals with graphs where all nodes are of the valence 4. One easy way to obtain such graphs is to draw one or more, self- and mutually intersecting closed curves, – or by projecting a knot or link (*KL*) into a plane. The paper now deals with these projected curves or shadows, trying to find an Eulerian path that visits all edges exactly once, but that never crosses itself.

If we put a two-sided mirror in every vertex of a *KL* shadow (ADAMS, 1994), by a suitable choice of mirror positions, we can obtain a single closed mirror curve (JABLAN, 2001). The choice of mirror positions is made according to the rules for obtaining a single mirror curve. This kind of mirror curve is a self-avoiding path, dividing the plane R^2 or sphere surface S^2 into two regions (interior and exterior). On a sphere, those two regions are equivalent.

Figure 3 shows two self-avoiding curves derived from the edge figure of an octahedron (that from knot theory point of view represents a shadow of Borromean rings), and a self-avoiding curve derived from the fullerene C_{60} by the mid-edge truncation. The number of mirrors of each kind necessary to convert any minimal *KL* shadow of a given *KL* into a self-avoiding curve is an invariant of the *KL*. For both self-avoiding curves derived from Borromean rings, the corresponding number of mirrors is $\{3, 3\}$ (3 internal and 3 external mirrors).

Fig. 1. Sculpture “Lombard” by Carlo Sequin.

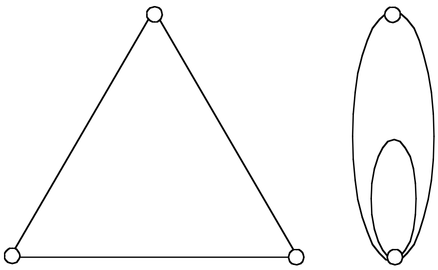

Fig. 2. Triangle with two vertices.

If we denote mirror points of a self-avoiding curve by 1, 2, ..., $2n$, we conclude that every self-avoiding curve is an $2n$-gon with n pairs of combined points. In order to avoid loops, we never collapse adjacent points. If we denote points belonging to internal mirrors by overlined numbers, and underline numbers belonging to external mirrors, the first self-avoiding curve (Fig. 4) can be denoted by the code $\overline{1}, \underline{2}, \underline{3}, \overline{4}, \overline{5}, \underline{6}, \underline{7}, \overline{8}, \overline{9}, \underline{10}, \underline{11}, \overline{12}$.

Analogously to Gauss codes of *KL*s (MURASUGI, 1996), this code will depend on a beginning point and orientation. Hence, several codes can be assigned to every self-avoiding curve and we can choose the minimal one as a representative. If we think of the exterior and interior of a curve as equivalent, then a code and its dual (the code with inverted overlinings and underlinings) are considered to be the same. In the same way as with Gauss and Dowker codes (ADAMS, 1994; MURASUGI, 1996), the proposed codes for self-avoiding curves can be written in a more concise form. First, we can write our code as a sorted list of ordered pairs $\{\{\overline{1}, \overline{4}\},\{\underline{2}, 1\underline{1}\},\{\underline{3}, 6\},\{\overline{5}, \overline{8}\},\{7, 10\},\{\overline{9}, \overline{12}\}\}$. If we agree to replace every pair of overlined numbers by the same numbers without overlinings, to replace every

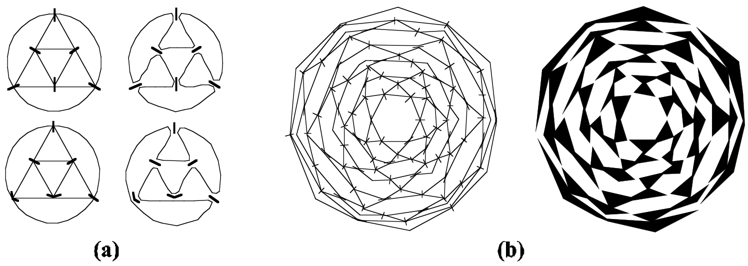

Fig. 3. (a) Two self-avoiding curves derived from the edge figure of an octahedron, that from knot theory point of view represents shadow of Borromean rings; (b) self-avoiding curve derived from the fullerene C_{60} by mid-edge truncation.

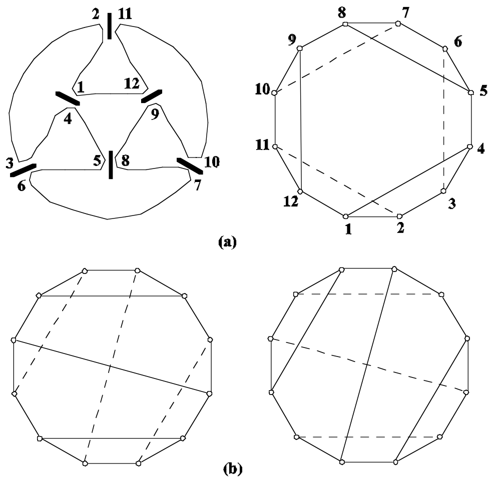

Fig. 4. (a) Coding of the first self-avoiding curve from Fig. 3a and its chord diagram; (b) chord diagram of the second self-avoiding curve from Fig. 3a and its dual.

pair of underlined numbers by those numbers in opposite (descending) order, and sort the obtained list, the result is the list {{1, 4},{5, 8},{6, 3},{9, 12},{10, 7},{11, 2}}. The list of second elements in each pair gives the short code {4, 8, 3, 12, 7, 2}. The complete code can be easily restored from the short code, knowing that the first parts of the ordered pairs are ordered numbers from the set {1, 2, ..., 12}, not belonging to the set {4, 8, 3, 12, 7, 2}, i.e., the numbers {1, 5, 6, 9, 10, 11}. A different agreement: replacing every pair of overlined numbers by those numbers in opposite (descending) order, every pair of underlined numbers by the same pair of numbers and sort the obtained list, gives the dual list {{2, 11},{3, 6},{4, 1},{7, 10},{8, 5},{12, 9}}, and the dual short code {11, 6, 1, 10, 5, 9}.

Every self-avoiding curve can be graphically interpreted by a *chord diagram*: a regular

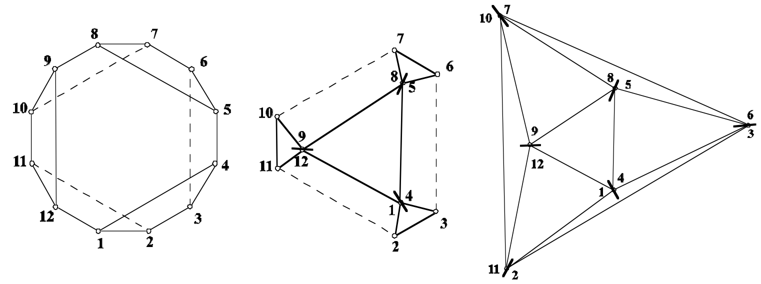

Fig. 5. Recovering of self-avoiding curve from its chord diagram.

2n-gon, where points belonging to internal mirrors are connected by full, and points belonging to external mirrors by broken diagonal lines (or by black and white lines). For example, the first self-avoiding curve from Fig. 3a will be described by the chord diagram from Fig. 4a, or by its dual obtained by inverse bicoloring, where full (black) lines are replaced by broken (white) lines and *vice versa*. In the both cases (Fig. 4) chord diagrams are equal to their duals, i.e., they are self-dual. From every chord diagram we can easily obtain a code of the corresponding self-avoiding curve and vice versa, and to recover an original *KL* from which that curve is derived. This is illustrated in Fig. 5, where two stages of the recovering are shown.

In order to derive and enumerate all self-avoiding curves with n mirrors, one should first derive all different non-colored chord diagrams, and then impose the appropriate coloring. The following rule holds for non-colored chord diagrams: every vertex belongs to exactly one diagonal (chord). In fact, we are searching for all different minimal sets of diagonals that span a regular 2n-gon, where sets that can be obtained one from another by symmetries of 2n-gon are considered to be the same. For n = 2, 3, ..., 7 we obtain, respectively 1, 2, 7, 29, 176, 1788 such different sets. The sequence obtained is A003437 from the On-line encyclopedia of integer sequences (http://www.research.att.com/~njas/sequences/), that represents the number of unlabeled Hamiltonian circuits on n-octahedron (SINGMASTER, 1975). An n-octahedron is the complete n-partite graph $K_{2,2,\ldots,2}$ (n pairs of opposite vertices with edges connecting each vertex to every other vertex except its opposite). D. Singmaster notes that such a Hamiltonian cycle can be viewed as a way of seating n couples around a circular table so that no man is next to his wife. The number of cases is given by the formula

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} \left[\frac{2n}{n-k}\right] 2^k (2n-k)!}{2^n n!}.$$

Chord diagrams derived for n = 2, 3, 4 are given in Fig. 6. Among all chord diagrams we can distinguish 2-vertex connected graphs (containing the edges of an 2n-gon as well), corresponding to non-prime *KL*s, and others, 3-connected, corresponding to prime *KL*s.

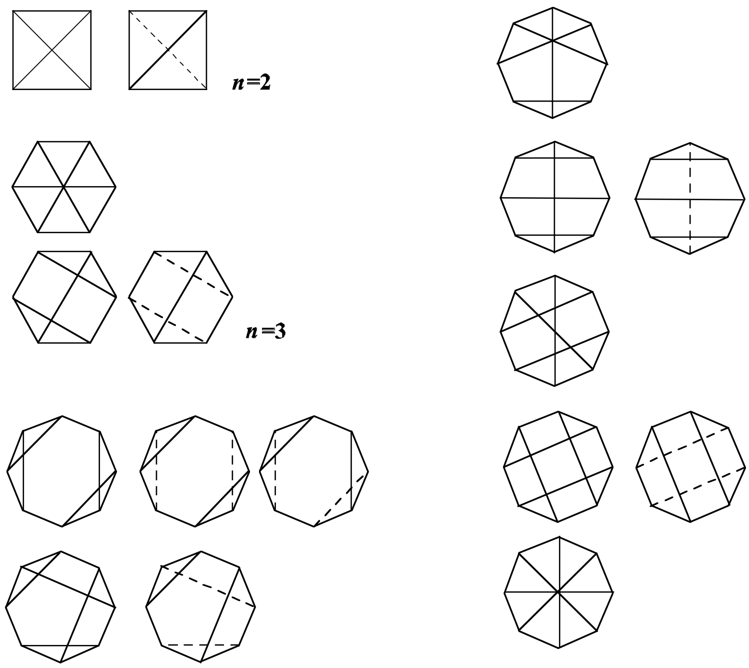

Fig. 6. Chord diagrams for $n = 2, 3, 4$.

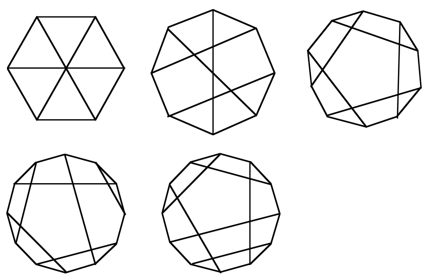

Fig. 7. Non-colorable chord diagrams.

For coloring of chord diagrams we have the rule: every two diagonals crossing each other must have different colors. A chord diagram will be colorable *iff* it is planar. The other, purely visual, criterion for colorability is the following: a chord diagram is colorable *iff* crossings of its diagonals do not form a polygon with an odd number of edges, and three or more diagonals have not a common point (Fig. 7). Coloring of a (colorable) chord diagram represents a projection of a polyhedron enclosed in an $2n$-gon, with proper visibility of all edges.

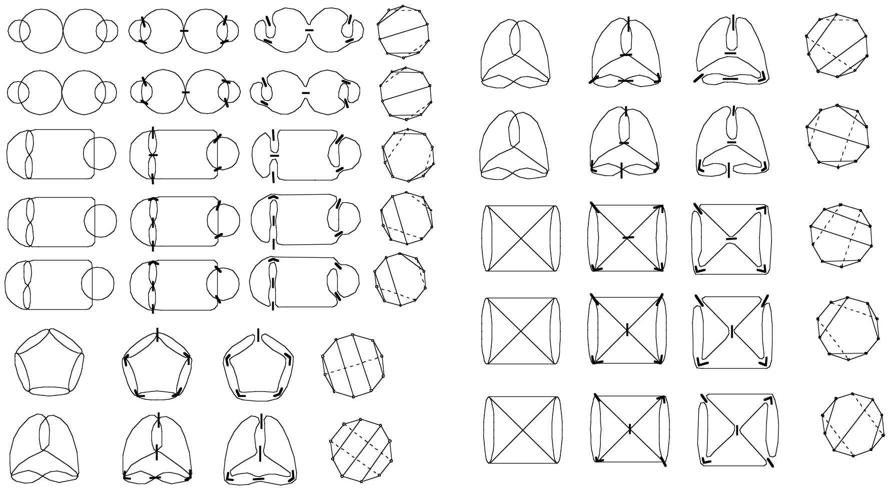

Fig. 8. *KL* shadows, self-avoiding curves, and colored chord diagrams obtained for n = 5.

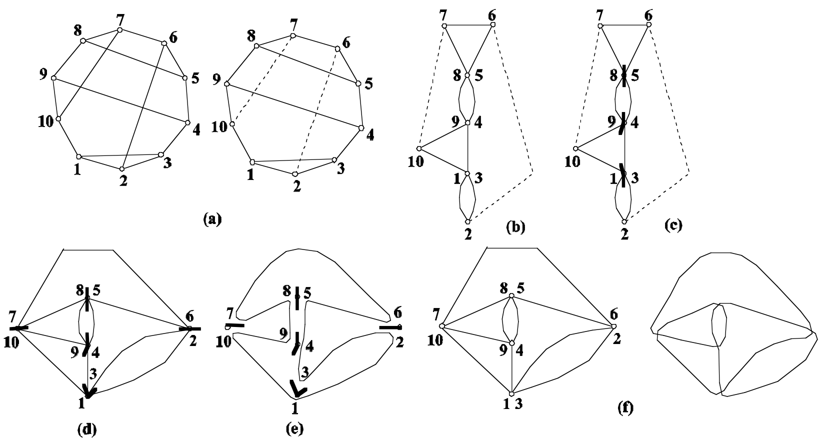

Fig. 9. (a) Uncolored chord diagram and its bicoloring; (b)–(e) reconstruction of its corresponding self-avoiding curve; (f) the corresponding *KL* shadow.

In the case of 2-vertex connected chord diagrams, coloring is not unique: from the same uncolored chord diagram we can obtain several different colored diagrams (Fig. 6). *KL* shadows, their corresponding self-avoiding curves and colored chord diagrams for n = 2, 3, 4 are given in Fig. 8. In the case of 3-vertex connected chord diagrams, a coloring is completely forced by the coloring of one edge: by choosing its color we can obtain only one colored chord diagram, or its dual. Hence, in the case of 3-vertex connected planar diagrams, an uncolored chord diagram provides complete information about the

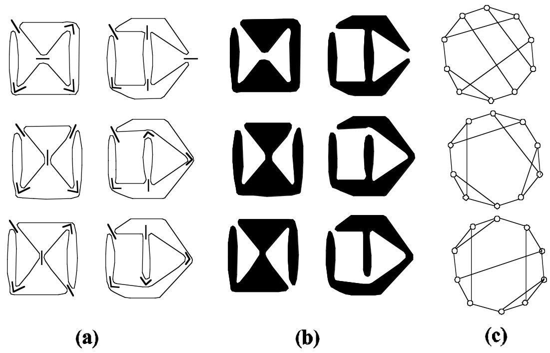

Fig. 10. Three pairs of equal self-avoiding curves (a) shown as shapes (b) and their chord diagrams (c).

corresponding self-avoiding curve. Every uncolored chord diagram can be given as a list of unordered pairs of numbers denoting chords. For example, the uncolored chord diagram from Fig. 9a can be denoted as {{1, 3},{2, 6},{4, 9},{5, 8},{7, 10}}. The same figure illustrates its bicoloring (a), the reconstruction of its corresponding self-avoiding curve (b)–(e), and *KL* shadow obtained (f).

By restricting our attention to 3-vertex connected planar chord diagrams corresponding to prime *KL*s, for n = 2, 3, ..., 8 we obtain, respectively, 1, 1, 3, 7, 33, 148, 923 chord diagrams corresponding to self-avoiding curves derived from prime *KL*s. For $n = 2$ we have one chord diagram {{1, 3},{2, 4}}, and for $n = 3$ one diagram {{1, 3},{2, 5},{4, 6}}. For $n = 3$ there are three diagrams: {{1, 3},{2, 5},{4, 7},{6, 8}}, {{1, 3},{2, 6}, {4, 8},{5, 7}}, and {{1, 4},{2, 7},{3, 6},{5, 8}}.

For $n = 5$, the seven chord diagrams are: {{1, 3},{2, 5},{4, 7},{6, 9},{8, 10}}, {{1, 3},{2, 5},{4, 8},{6, 10}, {7, 9}},{{1, 3},{2, 5},{4, 9},{6, 8},{7, 10}},{{1, 3},{2, 6},{4, 9},{5, 7},{8, 10}},{{1, 3},{2, 6},{4, 9},{5, 8}, {7, 10}}, {{1, 3},{2, 7},{4, 10},{5, 9},{6, 8}}, and {{1, 4},{2, 8},{3, 7},{5, 10},{6, 9}}.

For $n = 6$, there are thirty three such diagrams and their corresponding self-avoiding curves. Different shadows of the same *KL* can give different self-avoiding curves, as in the case of the link 2 2 2.

Visual recognition of self-avoiding curves, either direct or from shapes (Fig. 10b), is complicated even for a small number of mirrors, but it is almost immediate from chord diagrams (Fig. 10c).

It is interesting to mention a possible connection between shapes originating from self-avoiding curves and some biological forms.

From every *KL* shadow can be derived one or several self-avoiding curves. Some conclusions about original *KL*s can be made based on the chord diagrams of those self-avoiding curves. For example, to every diagonal connecting two vertices separated by one vertex, and to every pair of parallel adjacent diagonals corresponds a digon in the original *KL* shadow; diagrams without them correspond to basic polyhedra. This way, we can follow a process of digon collapsing directly in chord diagrams.

Among all chord diagrams, we can distinguish antisymmetrical diagrams that remain

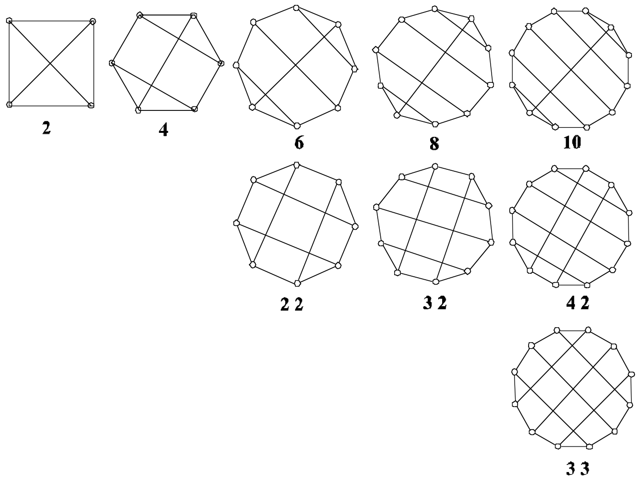

Fig. 11. Families of chord diagrams and self-avoiding curves.

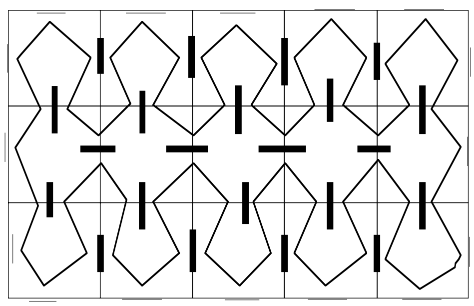

Fig. 12. *Via tori* that can be obtained by identifying opposite sides of the rectangle.

unchanged by opposite coloring of chords. With regard to self-avoiding curves, this means that the external region is equal to the internal one. Such diagrams are self-dual. For example, for $n = 6$, eleven among 33 chord diagrams are self-dual.

Again, families of *KL*s play important role as before, followed by families of chord diagrams and self-avoiding curves derived from them. Chord diagrams belonging to a same family can be visually recognized (Fig. 11).

Self-avoiding curves can be embedded on different surfaces, so together with *Viae Globi* on a sphere S^3, we can consider *Viae Tori* on a torus (later introduced in analogy to Sequin's *Viae Globi*), or on any other surface (Fig. 12).

The function **fDiffViae** from the knot theory computer program *LinKnot* (JABLAN and SAZDANOVI, 2006) for a given number *n* derives all different self-avoiding curves with *n* mirrors that can be obtained from prime *KL*s with *n* crossings. For every such curve given by its (uncolored) chord diagram we can find its basic prime *KL* by using the LinKnot function **fViaToKL**.

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